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The Problem of Mixing Meta-Language with Object Language in in Modal Systems

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<Abstract>

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Abstract:

In this paper, I discuss "the problem of language-level mixture"(PLLM) in modal semantics as follows: First, I explain the problem in detail. I choose propositional calculus and first-order logic, which are accepted as standard formal systems, and add some clauses to each of them in order to achieve seriously problematic systems: $\mathscr{P}^{\#}$ and $\mathscr{L}^{\#}$. Second, I analyze the exact points of these problematic systems and explain why they cannot be accepted. Third, I describe the formal systems of Carnap (1947), Kanger (1957), and Montague (1960) and show that each of these systems includes PLLM, at least partly.

I. Preface

In this paper, I discuss a problem called "the problem of language-level mixture" (PLLM) related to a defect in the construction of formal systems. PLLM refers to the problem of mixing a meta language and an object language in a faulty manner to construct some formal semantics. In this paper, I show that the modal systems of Carnap (1947), Kanger (1957), and Montague (1960) have PLLM.

The entire discussion presented in this paper can be divided into three parts: In the first part, I explain the specific meaning of PLLM. For this purpose, I modify the proposition calculus $\mathfrak{P}$ and the first-order logic $\mathfrak{L}$, which are usually considered to be the least problematic systems, to construct $\mathfrak{P}^{\#}$ and $\mathfrak{L}^{\#}$, which are the systems that have a serious problem, by a simple but evidently incorrect manipulation. In the second step, I point out the specific properties of systems that include PLLM and then explain why these systems cannot be accepted. In the third step, I show that the modal systems of Carnap, Kanger, and Montague include PLLM, by carefully describing these systems.

For the sake of an exact analysis and the convenience of understanding, I have defined the several consistent symbols used in this paper at the beginning. In this paper, I have used $\mathfrak{S}$, $\mathfrak{P}$, and $\mathfrak{L}$ to denote formal systems. Here, $\mathfrak{S}$ is used for denoting any of $\mathfrak{P}$ or $\mathfrak{L}$.[i] Further, a system having PLLM is denoted by the subscript "#," for instance, $\mathfrak{S}^{\#}$. Each italicized symbol is different from the non-italicized one, and every symbol having a different shape is regarded as a different symbol. Moreover, when a system is suggested as a sample, its entire syntax or all its axiomatic schemes or inference rules are omitted.

II. Systems having PLLM: $\mathfrak{P}^{\#}$ and $\mathfrak{L}^{\#}$

In order to show what PLLM is and how it can be realized, I have modified $\mathfrak{P}$ and $\mathfrak{L}$ to construct two faulty formal systems $\mathfrak{P}^{\#}$ and $\mathfrak{L}^{\#}$. As the syntax of $\mathfrak{P}$ and $\mathfrak{L}$ is well known and the focus is on the formal semantics of these systems, I have skipped the discussion of the standard syntax of these systems.

II.1. Semantics of Propositional Calculus $\mathfrak{P}$[ii]

A truth assignment v for a set S of sentence symbols is a function as follows:

$$v : S \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

Let $\overline{S}$ be the set of wffs generated from S by the formula-building operations syntactically defined. Then, an extension $\overline{v}$ of v can be expressed as follows:

$$\overline{v} : \overline{S} \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

$\bar{v}$ assigns the correct truth value to each wff in $\bar{S}$ as follows:

(II.1.0) For any $\alpha \in S$, $\bar{v}(\alpha) = v(\alpha)$.

Moreover, for any α, β in $\bar{S}$

(II.1.1) $\bar{v}((\neg\alpha))$ is **T** if $\bar{v}(\alpha) =$ **F**, and **F** otherwise.

(II.1.2) $\bar{v}((\alpha \rightarrow \beta))$ is **F** if $\bar{v}(\alpha) =$ **T** and $\bar{v}(\beta) =$ **F**, and **T** otherwise.

Now, consider a set Σ of wffs. For a simultaneous assignment v for all sentence symbols in every element of Σ, we say that "v satisfies Σ" iff every element of Σ is true in v.

(II.1.3) [Tautological implication: $\vDash$] Σ *tautologically implies* ϕ ($\Sigma \vDash \phi$) iff every v that satisfies every member of Σ also satisfies ϕ.

(II.1.4) [Tautology] ϕ is *tautology* iff $\emptyset \vDash \phi$. In general, we write "$\vDash \phi$" iff "$\emptyset \vDash \phi$." Hence, every v satisfies ϕ iff $\vDash \phi$.

II.2. Semantics of First-Order Logic $\mathcal{L}$

For the definition of the semantics of the first-order logic $\mathcal{L}$, let

δ, ϕ, ψ, etc. be wffs of $\mathcal{L}$,

$\mathfrak{A}$ be a structure (or interpretation) for $\mathcal{L}$ (The *universe* $|\mathfrak{A}|$ of $\mathfrak{A}$ is a nonempty set),

$s: V \rightarrow |\mathfrak{A}|$ be a function from the set V of all variables in the *universe* $|\mathfrak{A}|$ of $\mathfrak{A}$.

Then, for a sentence δ and structures $\mathfrak{A}$, we define

$$\vDash_{\mathfrak{A}} \delta$$

as "δ is true in $\mathfrak{A}$," and define

$$\vDash_{\mathfrak{A}} \phi[s]$$

as "$\mathfrak{A}$ satisfy ϕ with s." The formal definition of satisfaction is as follows:

(II.2.1) [Terms] The terms are defined as follows: First, we define the extension

$$\bar{s}: T \rightarrow |\mathfrak{A}|,$$

as a function from the set T of all terms in the universe of $\mathfrak{A}$. $\bar{s}$ is defined by recursion as follows:

1. For each variable x, $\bar{s}(x) = s(x)$.

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2. For each constant symbol c, $\bar{s}(c) = c^{\mathfrak{A}}$.

(II.2.2) [Atomic formulas] The definition of the satisfaction of atomic formulas is as follows:

For an n-place predicate parameter P,

$$\models_{\mathfrak{A}} Pt_1 \cdots t_n[s] \text{ iff } < \bar{s}(t_1), \cdots, \bar{s}(t_n) > \in P^{\mathfrak{A}}.$$

(II.2.3) [Other wffs] The satisfaction of the wffs defined inductively is defined recursively.

(1) For atomic formulas, the definition is given above.

(2) $\models_{\mathfrak{A}} \neg\phi[s]$ iff $\not\models_{\mathfrak{A}} \phi[s]$.

(3) $\models_{\mathfrak{A}} (\phi \to \psi)[s]$ iff $\not\models_{\mathfrak{A}} \phi[s]$ or $\models_{\mathfrak{A}} \psi[s]$ or both.

(4) $\models_{\mathfrak{A}} \forall x\phi[s]$ iff for every $d \in |\mathfrak{A}|$, we have $\models_{\mathfrak{A}} \phi[s(x|d)]$.

Here, $s(x|d)$ is the function that is exactly the same as s except for one difference: At the variable x, it assumes the value d. This can be expressed by the following equation:

$$s(x|d)(y) = \begin{cases} s(y) & if \quad y \neq x \\ d & if \quad y = x \end{cases}$$

(II.2.4) [Logical implication: $\models$] Let Γ be a set of wffs and ϕ a wff. Then, Γ logically implies ϕ ($\Gamma \models \phi$) iff for every structure $\mathfrak{A}$ for the language and for every function $s: V \to |\mathfrak{A}|$, such that $\mathfrak{A}$ satisfies every member of Γ with s, $\mathfrak{A}$ also satisfies ϕ with s.

(II.2.5) [Validity] ϕ is *valid* iff $\emptyset \models \phi$. In most cases, "$\emptyset \models \phi$" is written simply as "$\models \phi$." Thus,

$\models \phi$ iff for every $\mathfrak{A}$ and every $s: V \to |\mathfrak{A}|$, $\mathfrak{A}$ satisfies ϕ with s.

II.3. Problem of Language-Level Mixture

Next, I make some simple modifications to the syntax and semantics of $\mathfrak{P}$ and $\mathfrak{L}$ to construct new systems $\mathfrak{P}^{\#}$ and $\mathfrak{L}^{\#}$, which will clearly indicate what PLLM is. I believe that for any system $\mathfrak{S}$, PLLM in $\mathfrak{S}^{\#}$ may seem to be non-problematic at the first glance.[iii] Therefore, let me first construct $\mathfrak{S}^{*}$ that will show PLLM intuitively and clearly, and then modify it to $\mathfrak{S}^{\#}$.

$\mathfrak{P}^{*}$ can be obtained by adding the following clauses.

(II.3.1) [Syntax of $\mathfrak{P}^{*}$] For a wff ϕ of $\mathfrak{P}$, $\models \phi$ is also a wff.

(II.3.2) [Semantics of $\mathscr{P}^*$] $\bar{v}(\models\phi)=\mathbf{T}$ iff every v satisfies ϕ.

Here, the right side of (II.3.2) is the same as the right side of (II.1.4), which is the definition of tautology. Then, we can combine (II.3.2) with (II.1.4) and deduce the following:

(II.3.3) $\bar{v}(\models\phi)=\mathbf{T}$ iff $\models\phi$.

It is intuitively reluctant for one to accept this system. Then, perhaps we can substitute the symbol "□" for "$\models$" in the syntax of $\mathscr{P}^*$ to construct the system $\mathscr{P}^\#$ as follows (Here, only one $\models$ of two in (II.3.3) is substituted by □.)

(II.3.4) [Syntax of $\mathscr{P}^\#$] Add □ to the set of primitive symbols of $\mathscr{P}^\#$. Further, for a wff ϕ, $\Box\phi$ is also a wff.

(II.3.5) [Semantics of $\mathscr{P}^\#$] $\bar{v}(\Box\phi)=\mathbf{T}$ iff every v satisfies ϕ.

Then, (II.3.3) will be modified as follows in $\mathscr{P}^\#$.

(II.3.6) $\bar{v}(\Box\phi)=\mathbf{T}$ iff $\models\phi$.

On the other hand, $\mathscr{L}^\#$ will be constructed in a similar manner. First, we add the following to $\mathscr{L}$ and construct $\mathscr{L}^*$.

(II.3.7) [Syntax of $\mathscr{L}^*$] For a wff ϕ of $\mathscr{L}$, $\models\phi$ is also a wff.

(II.3.8) [Semantics of $\mathscr{L}^*$] $\models_{\mathfrak{A}}(\models\phi)$[iv] iff, for every structure $\mathfrak{A}$ and every $s: V\rightarrow|\mathfrak{A}|$, $\mathfrak{A}$ satisfies ϕ with s.

Here, the right side of (II.3.8) is the same as the right side of (II.2.5), which is the definition of validity. Then, we can combine (II.3.8) with (II.2.5) and deduce the following:

(II.3.9) $\vDash_{\mathfrak{A}}(\vDash \phi)$ iff $\vDash \phi$.

Then, we can find a problem of $\mathfrak{L}^*$ similar to that of $\mathfrak{P}^*$. Now, it is intuitively reluctant for one to accept this system. Hence, we substitute the symbol "□" for "$\vDash$" in the syntax of $\mathfrak{L}^*$ and construct $\mathfrak{L}^\#$. Now, $\mathfrak{L}^\#$는 is the system of the first-order logic with (II.3.10) and (II.3.11) instead of (II.3.7) and (II.3.8), as follows:

(II.3.10) [Syntax of $\mathfrak{L}^\#$] Add □ to the set of primitive symbols of $\mathfrak{L}^\#$. Further, for a wff ϕ, $\Box\phi$ is also a wff.

(II.3.11) [Semantics of $\mathfrak{L}^\#$] $\vDash_{\mathfrak{A}}\Box\phi$ iff for every structure $\mathfrak{A}$ and every $s: V \rightarrow |\mathfrak{A}|$, $\mathfrak{A}$ satisfies ϕ with s.

On the other hand, (II.3.9) of $\mathfrak{L}^*$ will be modified as follows in $\mathfrak{L}^\#$.

(II.3.12) $\vDash_{\mathfrak{A}}\Box\phi$ iff $\vDash \phi$.

Now, considering the cases of $\mathfrak{P}^\#$ and $\mathfrak{L}^\#$, the following can be stated: PLLM is clearly present in $\mathfrak{MS}^*$, and actually, $\mathfrak{MS}^\#$ has the same problem (PLLM). Although instead of the symbol of the meta language $\vDash$, the new symbol □ is present in the syntax of $\mathfrak{MS}^\#$, the previous statement is true. We can observe this, because the definitions of (II.3.2) and (II.3.5) as well as those of (II.3.8) and (II.3.11) reveal that $\vDash$ in $\mathfrak{MS}^*$ and □ in $\mathfrak{MS}^\#$ are the same, despite their different notations.

The process to realize PLLM in $\mathfrak{MS}^\#$ can be analyzed in the following three steps:

The first step can be seen in (II.3.1) and (II.3.7). In these steps, the symbol "$\vDash$," which can be used only in the semantics, was used for forming a wff without any modification of its meaning in $\mathfrak{MS}$. In short, the symbol of the meta language was mixed into the object language.

The second step was to define the truth conditions in (II.3.2) and (II.3.8), where the meaning of the syntactical symbol "$\vDash$" is equal to that of the semantical symbol "$\vDash$" in $\mathfrak{MS}^*$. Two syntactical symbols must be the same when their truth conditions are the same, even if

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they have different notations. Hence, the “⊨” symbols in both the syntax and the semantics are exactly the same.

PLLM, which can be seen in these two steps, results in (II.3.6) and (II.3.12), and this is the third step. However, at the first glance, (II.3.6) and (II.3.12) may seem to fit Tarski’s (1944) T-scheme (“x is true iff P”). However, this is not the case in reality. It is the truth condition of the atomic sentence in (II.2.2) that fits the T-scheme well. In the left side of (II.2.2), “$Pt_1 \cdots t_n$” is the name of a sentence, which is combined with “$\vDash_{\mathfrak{A}}$ (⋯is true)” to form the entire left side. On the other hand, “$< \bar{s}(t_1), \cdots, \bar{s}(t_n) > \in P^{\mathfrak{A}}$” is the entire right side, and this is not the name of the sentence, but what the name of the sentence denotes. If we examine the truth conditions of (II.3.6) and (II.3.12) on the basis of these considerations, we can see that their content follows the form of “x is true iff x.” This is the key property of a semantically closed system, which Tarski diagnosed as the cause of a liar’s paradox.

III. Logical Properties of a System 𝔖# where PLLM Occurs

Using the examples of II, I believe that I have sufficiently explained what I mean when I say that PLLM occurs in a formal system 𝔖 and what can be a typical process where it occurs. Next, I will demonstrate that this problem occurs in some modal formal systems. Moreover, I want to show that if 𝔖 has PLLM, 𝔖 cannot be a formal system that is constructed in the right manner. Then, because I also need to clearly state why this is so, ① I check the basic properties of the formal semantics of 𝔖 and its basic formation rules; ② I examine how PLLM in 𝔖* or 𝔖# breaks these rules and what logical problem follows from it; and ③ I clarify the properties on the basis of which we can select the problematic system 𝔖#. Thus, the information in ③ should be useful as a yardstick to test whether 𝔖 has PLLM for any system 𝔖.

III.1. Basic Properties of Formal Semantics of 𝔖 and Its Basic Formation Rules

First, let me discuss several basic principles by which we understand and construct model-theoretic semantics. These principles are applied to the definitions of every model-theoretic semantics in such a natural manner that they are rarely discussed or described clearly. However, we need to have them clarified for the purpose of analyzing problems of $\mathcal{QS}^{\#}$ suggested in II.

For this, I need to determine what can be the appropriate functions of a model or a structure in model-theoretic semantics. I discuss the case of $\mathcal{L}$ for specific consideration. The structure $\mathfrak{A}$ of $\mathcal{L}$ in Ch. II is sometimes also referred to as model[v], and if we reflect on the case of a particular sentence Fab of $\mathcal{L}$ and the interpretation $\mathfrak{A}$ given for it, the semantical relation can be given by the following table (III.1.0).

Table (III.1.0)

Models	$\mathfrak{A}_1$	$\mathfrak{A}_2$	⋯
$\lvert\mathfrak{A}\rvert$	t,u,v	$\{\alpha, \beta, \gamma\}$	
F	<t,t>, <t,u>, <t,v>	$<\alpha, \beta>, <\beta, \gamma>$	⋯
a	t	α	⋯
b	u	γ	⋯

It is easy to understand what table (III.1.0) conveys. The first left column contains the universe, particular parameters, and the rest of the columns show several examples of the specific interpretations. Thus, a refers to t and b to u in the interpretation $\mathfrak{A}_1$. In a certain column where the top item is $\mathfrak{A}_n$, we can also assign an interpretation "(in the actual world)⋯ is the father of ~" to F, Tom to a, and Jerry to b.

Here, I want to emphasize two facts that are very clearly accepted as part of the general knowledge of a formal system.

First, each model of $\mathcal{L}$ provides distinct conditions for "any" wff ϕ to be true or false. These are truth conditions. For example, Fab and $\forall xFax$ are verified as true in $\mathfrak{A}_1$. In $\mathfrak{A}_2$, Fab and $\forall xFax$ are false.

Second, $\lvert\mathfrak{A}\rvert$ of $\mathcal{L}$ can be $\lvert\mathfrak{A}_1\rvert$ or $\lvert\mathfrak{A}_2\rvert$, but it cannot be both $\lvert\mathfrak{A}_1\rvert$ and $\lvert\mathfrak{A}_2\rvert$ simultaneously. Moreover, this type of model, i.e., $\lvert\mathfrak{A}\rvert$, is not possible in $\mathcal{L}$, which includes several (more than one) $\lvert\mathfrak{A}_i\rvert$ s that are enumerated in a row.

Although I have provided an explanation using an example of $\mathcal{L}$, I believe that these two demands should be satisfied not only for $\mathcal{L}$ but also for every formal system in general. To be exact, for any $\mathcal{OS}$, each model of $\mathcal{OS}$ should be able to provide the truth condition for any wff ϕ in it and each individual model $\mathfrak{A}_i$ cannot include another equivalent individual model $\mathfrak{A}_j$. I believe that these two demands describe some basic principles that we generally accept whenever we construct model-theoretic semantics.

This, of course, does not exclude the possibility that there can be such a system as $\mathcal{OS}+$, where a truth condition for a sentence ψ can demand several models. For example, it can be possible that a set of some model $\boldsymbol{m}$s is assigned to $|\mathfrak{A}|$ in $\mathfrak{A}_n$ of $\mathcal{OS}_1$. However, in this case, a cell of a row of $\mathfrak{A}_n$ should contain { $\boldsymbol{m}$ | $\boldsymbol{m}$ is …}, and it should be a starting point for the construction of the model. Simultaneously, $\boldsymbol{m}$ cannot be equivalent to a $\mathfrak{A}_m$ that appears in the row of $\mathfrak{A}_n$.[vi]

On the other hand, we can infer some basic principles of constructing formal semantics on the basis of the concept of a "function."

The interpretation $\boldsymbol{i}$ for system $\mathcal{OS}$ is called "a structure," and $\boldsymbol{i}$ is a function (Enderton 1972, p. 79). You can understand this fact easily if you reflect on the basic object for the construction of the interpretation $\boldsymbol{i}$. The interpretation $\boldsymbol{i}$ is designed to assign a univocal meaning to each syntactical sign in a formal system $\mathcal{OS}$. An interpretation $\boldsymbol{i}$ should assign neither several meanings nor none to a syntactical sign or sentence. This property of $\boldsymbol{i}$, therefore, exactly satisfies the definition of a function.

The interpretation $\boldsymbol{i}$ for $\mathcal{P}$ is $\bar{v}$ in the interpretation II.1. The argument of $\bar{v}$ is wff of $\mathcal{P}$, and its value is a truth value. This means that dom($\bar{v}$) = $\bar{S}$ and ran($\bar{v}$) = {**T**, **F**}. The interpretation $\boldsymbol{i}$ for $\mathcal{L}$ is more complicated. $\mathfrak{A}$ constructs the references (i.e., values) that are assigned to some parts, and then $\bar{s}$ assigns the former to the latter. The function, which assigns the truth value to wff ϕ at the end, is not seen explicitly in the clause II.2. Instead, the symbol "$\models$" is used. Thus, you can see that the ordered pair <$\mathfrak{A}$, $\bar{s}$, ϕ> is the argument of $\models$, and its value is the truth value in (II.2.2) and (II.2.3). Let this function be $f_{\models}$; then, Enderton identifies the sum of $\bar{s}$ and $f_{\models}$ with $\mathfrak{A}$. From this viewpoint, dom($\mathfrak{A}$) is a set of parameters, and ran($\mathfrak{A}$) is as follows:[vii]

(III.1.1) $\mathfrak{A}$ assigns to the quantifier symbol $\forall$ a nonempty set $|\mathfrak{A}|$, called the *universe* of $\mathfrak{A}$.

(III.1.2) $\mathfrak{A}$ assigns to each n-place a predicate symbol P and an n-ary relation $P^{\mathfrak{a}} \subseteq |\mathfrak{A}|^n$; i.e., $P^{\mathfrak{a}}$ is a set of n-tuples of the members of the universe.

(III.1.3) $\mathfrak{A}$ assigns to each constant symbol c a member $c^{\mathfrak{a}}$ of the universe $|\mathfrak{A}|$.

In a relatively brief summary, a structure $\mathfrak{A}$ for the first-order language tells us the following (Enderton 1972, p. 79):

(III.1.4) The collection of things that the universal quantifier symbol ($\forall$) refers to, and

(III.1.5) What the other parameters (the predicate and function symbols) denote.

In $\mathfrak{A}$, $\neg$, $\rightarrow$, and quantifiers are assigned their usual meanings, and this runs as is defined in (2), (3), and (4) of (II.2.3).

In summary, the interpretation $\mathfrak{i}$ for a system $\mathfrak{S}$ is a function. Thus, every function f is a relation that assigns only an element of the range of f (ran(f)) to each element of the domain of f (dom(f)), which is a set of ordered pairs. Therefore, we can think of dom($\mathfrak{i}$) and ran($\mathfrak{i}$) for an interpretation $\mathfrak{i}$, which is a function. Here, $\mathfrak{i} \notin$ dom($\mathfrak{i}$) and $\mathfrak{i} \notin$ ran($\mathfrak{i}$) simultaneously. If it is possible that $\mathfrak{i} \in$ dom($\mathfrak{i}$) or $\mathfrak{i} \in$ ran($\mathfrak{i}$), a very serious problem occurs. Consider only the case of $f \in$ ran(f).[viii] Then, dom(f)={a, b, ⋯ } and ran(f)={f_1, f_2, ⋯ }. This makes $f = \{<a, f_1>, <b, f_2>, \cdots\}$, and when $f = f_1$, f will be as follows:

(III.1.6) $f = \{<a, \{<a, f_1>, <b, f_2>, \cdots\}>, <b, f_2>, \cdots\}$

The right side of (III.1.6) includes f_1 again. Hence, it can be substituted by the complete definition of f. This can be repeated forever. In conclusion, f cannot be defined when $f \in$ ran(f).

Moreover, $\forall f \notin$ ran(f) is not less than $f \notin$ ran(f). Here, "$\forall f$" denotes "all f," i.e., the set of f. It is intuitively easy to understand that $\forall f \in$ ran(f) is logically problematic as far as $f \in$ ran(f) is logically problematic. In the case that $\forall f \in$ ran(f), at least one element of f must be $<a, \forall f>$, which will cause exactly the same problem as that observed when $f \in$ ran(f).

Therefore, we can extend the case of $\mathfrak{i} \in$ ran($\mathfrak{i}$) and state that if PLLM means $\forall \mathfrak{i} \in$ ran($\mathfrak{i}$)

in $\mathscr{S}^{\#}$, it means $\mathscr{S}^{\#}$ is a system that cannot be defined.

III.2. Condition for PLLM to Occur in System $\mathscr{S}^{\#}$

How can we recognize PLLM if there is a possibility that it is present in a certain system?

The logical properties designed by me for $\mathscr{T}^{\#}$ and $\mathscr{L}^{\#}$ can be clearly seen in (II.3.4), (II.3.5), (II.3.10), and (II.3.11). At least in my design, they are the following two logical properties that $\mathscr{S}^{\#}$ possesses.

(III.2.1) Each model of $\mathscr{S}^{\#}$ cannot provide a truth condition of some wff($\Box\phi$) of $\mathscr{S}^{\#}$.

(III.2.2) In $\mathscr{S}^{\#}$, the truth condition of some wffs includes the quantification of a metalinguistic function such as the assignment v or s, or structure $\mathfrak{A}$.

Let me discuss the property (III.2.1) first. It will be easy to understand why this property makes a formal system incorrect by checking what was discussed in III.1. This property disobeys the general principles along which we construct model-theoretic semantics. According to the concept of the model of formal semantics, a model should be defined in an abstract form such as $\langle D, V \rangle$, which can cover almost every concrete model. However, a truth condition for a specific sentence should ultimately be based on each concrete model. The truth condition for the sentence "Tom(c) and Jerry(a) are friends(F)" (Fca) is determined in a model(for example, the actual world) that assigns someone to "Tom" and another person to "Jerry" and a certain relation to "… are friends." It is definitely not the case that several models should be considered in order to determine the truth value of a sentence.

Next, let me discuss the implication of (III.2.2). (III.2.2) can be modified to (III.2.3) as follows:

(III.2.3) In $\mathscr{S}^{\#}$, something that includes the quantification on value assignment v or model $\mathfrak{A}$, which is a pure semantical entity, is assigned to at least one syntactical symbol.

(III.2.2) is equivalent to (III.2.3), because in a well-constructed formal system $\mathscr{S}$, each syntactical element of $\mathfrak{S}$ will have no more meaning than the explicit meaning(reference) defined (assigned) in a precise and clear manner.[ix] Let us suppose that for two sentences ϕ and $\Box\phi$ of $\mathfrak{S}$, the truth condition for ϕ does not include the quantification on the assignment or structure, while that of $\Box\phi$ includes the quantification on the assignment or structure. A truth condition assigns to each sentence a condition that makes it true or false. Thus, the truth condition is a function. As in the case of Tarski's (1944) scheme T ("x is true iff P"), it is evident that the definition of the truth condition(T) is a function that assigns P to x. Then, the content of the truth condition for $\Box\phi$ minus that of ϕ will be assigned to $\Box$. This implies that the quantification of the assignment or structure must belong to that which is assigned to $\Box$. Otherwise, a syntactical symbol of $\mathfrak{S}$ would have some other meaning than that defined or lack some meaning. In this case, $\mathfrak{S}$ would be a badly formulated system.

By recalling the discussion in III.1, it is easy to understand why the property of (III.2.3) causes some serious problems in the system $\mathfrak{S}^{\#}$. The reason is that (III.2.3) implies $\forall \boldsymbol{i} \in \text{ran}(\boldsymbol{i})$. Consider an actual example of (II.3.5). Its right side, which is a truth condition, is $\forall v v(\phi) = \mathbf{T}$, and this is a second-order sentence that universally quantifies the assignment v. Thus, the entire right side, except ϕ, must be assigned to $\Box$, which means that "every v satisfies …." That is, a function "$\forall v v(\cdots)$".[x] (II.3.8) can be analyzed in the same manner.

Let me discuss one more concept, which can be an anticipated refutation of my current analysis on PLLM. It is an argument that the meta language and the object language can be used together appropriately in some cases. In Tarski-style (1944) model-theoretic semantics, every sentence that occurs in the object language should also occur in the meta language; the meta language should include the object language as a part(Van Fraasen 1971). Hence, although the meta language and the object language are mixed in a system, this mixture itself does not result in the faulty use of the meta language and the object language. This argument can be right. However, PLLM does not indicate that the object language occur in the meta language, but that the meta language is a part of the object language. In particular, in PLLM, a predicate of the meta language is defined as that of the object language. The meta language is a part of the object language in (II.3.1), (II.3.2), (II.3.7), and (II.3.8) of the specific example (III.2.2), and this clear problem was hidden in the process of (II.3.4), (II.3.5), (II.3.10), and (II.3.11). What kind of manipulation was required to conceal this

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problem? The symbol "$\vDash$" is actually the same both in syntax and in semantics, whereas only the symbol of syntax was substituted by $\Box$. It is a faulty formal manipulation to substitute a symbol($\Box$) for another($\vDash$) only in the syntax. Hence, this faulty manipulation leads to the observation that $\mathfrak{S}^{\#}$ can be appropriately formulated when it is not suggested by $\mathfrak{S}^{*}$.

In summary, the occurrence of (III.2.1) or (III.2.2) make it impossible to correctly define a formal system as far as any of both occurs in the system. This implies that either (III.2.1) or (III.2.2) can be a criterion for the presence of PLLM in a system $\mathfrak{S}$.

Now, let me analyze some formal systems that have PLLM on the basis of the mentioned understanding.

IV. PLLM in Carnap' s Modal Logic System

The first system I examine here is the modal logic system of Carnap (1947). Carnap was the first philosopher who introduced the model-theoretic method for modal logic (Feys 1963; Hintikka 1961, p. 292). Let me divide Carnap's modal system into propositional modal logic(CPML) and quantified modal logic (CQML) for a detailed discussion.

IV.1. Syntax and Semantics of Carnap's Propositional Modal Logic

CPML can be described using the modern notation as follows:

(IV.1.1) Syntax of CPML (Wffs and Formation Rules)

The set of the primitive symbols of CPML is that of $\mathfrak{P}$ with one addition: $\Box$.

The formation rules of CPML are as follows:

a. Propositional letter P is a wff.

b. If ϕ and ψ are wffs of CPML, $\neg\phi$ and $\phi \rightarrow \psi$ are wffs of CPML.

c. If ϕ is a wff of CPML, $\Box\phi$ is also a wff.

(IV.1.2) Semantics of CPML

CPML is constructed by extending $\mathfrak{P}$. Thus, the semantics for CPML is also constructed by extending the semantics for $\mathfrak{P}$. The starting point of the semantics of $\mathfrak{P}$ is the definition of truth. The definition of truth in $\mathfrak{S}$ is based on the concept of "state description." State-description σ is a set of propositional letters, "which contains, for every atomic sentence, either this sentence or its negation, but not both, and no other sentences (Carnap 1947, p. 9)."

There is one and only one true state description, which describes the actual state of the universe. Let me call it "@." @ contains all true atomic sentences and the negations of false ones. Hence, @ contains only true sentences (Carnap 1947, p. 10). Therefore, for an atomic sentence ϕ, ϕ is true iff $\phi \in @$. Based on these concepts, the semantics of every wff of $\mathfrak{S}$ is defined as follows:

(IV.1.2.1) ϕ is true. $=_{Df}$ $\phi \in @$.

(IV.1.2.2) $\phi \rightarrow \psi$ is true. $=_{Df}$ $\phi \notin @$ or $\psi \in @$.

(IV.1.2.3) $\neg\phi$ is true. $=_{Df}$ $\phi \notin @$.

These concepts produce Carnap modal logic (CML) systems, which include both CPML and CQML.[xi] Carnap himself did not distinguish CPML from CQML in his own writing, but I will distinguish them in my discussion.

CPML is obtained by adding a new operator "□" to its semantical definitions, and Carnap's semantics of CPML is as follows (Carnap 1947, pp. 10–11):

(IV.1.2.4) [Definitions] A sentence ϕ_i is L-true (in $\mathfrak{S}$). $=_{Df}$ ϕ_i holds in every state description (in $\mathfrak{S}$).

a. ϕ_i is L-false (in $\mathfrak{S}$). $=_{Df}$ $\neg\phi_i$ is L-true.

b. ϕ_i L-implies ϕ_j (in $\mathfrak{S}$). $=_{Df}$ $\phi_i \rightarrow \phi_j$ is L-true.

c. ϕ_i is L-equivalent to ϕ_j. $=_{Df}$ $\phi_i \equiv \phi_j$ is L-true.

d. ϕ_i is L-determinate (in $\mathfrak{S}$). $=_{Df}$ ϕ_i is L-true or L-false.

Next, Carnap's definition of the meaning of the modal operator symbol is as follows:

(IV.1.2.5) $\Box\phi$ is true iff ϕ is L-true.

(IV.1.2.6) $\Box\phi$ is L-true in CML iff ϕ is L-true.

IV.2. Syntax and Semantics of Carnap's Quantified Modal Logic System

Now, let me introduce CQML. The following is the CQML by Gheerbrant and Mostowski (2006, pp. 87–88).

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(IV.2.1) <Syntax of CQML>

In CQML, let σ be a purely relational vocabulary and $\mathfrak{B}=\{x_0, x_1, x_2, \cdots\}$ be the set of all the first-order variables.

(IV.2.1.1) [Definition] First, we define *the set* $AFrm_\sigma$ *of atomic formulas of vocabulary* σ :

$AFrm_\sigma = \{ x_i = x_j \mid i, j \in \omega \} \cup \{ P(x_{i_1}, x_{i_2}, \cdots, x_{i_n}) \mid i_1, i_2, \cdots, i_n \in \omega$, where P is a predicate of arity n in σ .}

$LFrm_\sigma$ is the smallest set X containing $AFrm_\sigma$ such that if $\phi, \psi \in X$, then

1. $\neg\phi \in X$,
2. $(\phi \rightarrow \psi) \in X$,
3. $\Box\phi \in X$, and
4. $\forall x_i \phi \in X$, for each $i \in \omega$.

(IV.2.2) <Semantics of CQML>

The formal semantics of CQML can be defined as follows.

(IV.2.2.1) [Definition] Let M be the σ -model. Any function $\bar{a} : \mathfrak{B} \rightarrow |M|$ is called *a valuation in* M . We define *the satisfaction relation of a formula* ϕ *in a model* M *under a valuation* $\bar{a}$ in the following manner:

(IV.2.2.2) For $\phi \in AFrm_\sigma$, $M \vDash \phi[\bar{a}]$ is defined in the same manner as for the first-order case.

(IV.2.2.3) For arbitrary $\phi, \psi \in LFrm_\sigma$, we have

(i) $M \vDash \neg\phi[\bar{a}] =_{Df} M \nvDash \phi[\bar{a}]$.

(ii) $M \vDash (\phi \rightarrow \psi)[\bar{a}] =_{Df}$. if $M \vDash \phi[\bar{a}]$, then $M \vDash \psi[\bar{a}]$.

(iii) $M \vDash \forall x_i \phi[\bar{a}] =_{Df} M \vDash \phi[\bar{a}(x_i / b)]$, for each $b \in |M|$.

(iv) $M \vDash \Box\phi[\bar{a}] =_{Df}$ for each σ -model M' , such that $|M| = |M'|$, $M' \vDash \phi[\bar{a}]$.[xii]

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IV.3. PLLM in Carnap's Modal System

Carnap did not define formal semantics for modal logic with the precision that we demand for a common formal system today. Hence, there might be room for us to attempt several different analyses on Carnap's modal semantics. Hence, let me briefly analyze PLLM based on (IV.2.2) first, and then show that Carnap's system may have PLLM on the basis of the semantics provided by Carnap himself.

First, between a state-description σ and a set Σ of σ, which could be the semantical model(or a structure) in CPML and CQML? It seems that he supposed that it is not Σ but σ that corresponds to v in $\mathscr{P}$, and $\mathfrak{A}$ in $\mathscr{L}$. Then, it can be undoubtedly declared that the CPML and CQML systems have PLLM when they are based on (III.2.1).

Next, it can still be asserted that Carnap's system has the same problems even when it is understood on the basis of (III.2.2), despite the small possibility of another interpretation. According to (IV.2.2), (iv) of (IV.2.2.3) clearly includes the quantification of model M'. As pointed out in (III.2.2), for the sake of preventing PLLM in a system, the truth condition of a sentence should include neither the reference nor the quantification of the assignment or model. In other words, the entire semantics of (IV.2.2) was constructed in the same manner as $\mathscr{L}^{\#}$. Both were constructed by first defining model M for the first-order logic and then considering "the quantification of M'" as the basis of the truth condition for the modal sentence $\Box\phi$.

Then, what if we analyze Carnap's system on the basis of not the semantics of CQML as Gheerbrant and Mostowski (2006) summarized, but on the basis of Carnap's semantics? I think that nothing should change even in this case.

Let me suppose that Carnap's semantics is based on a set of state-description σs. Then, (IV.1.2.1) in IV.1 corresponds to the definition of v that assigns the truth value to each sentential letter in $\mathscr{P}$. Formally, this can be expressed as follows:

(IV.3.1) $v(\phi) = \mathbf{T}$ iff $\phi \in @$.

$@ \in \Sigma$, and hence, we can generalize this definition (IV.3.1) to every σ, which is an element of Σ. Let me assign a unique v_i to each σ_i, and then define the following:

(IV.3.2) $v_i(\phi)=\mathbf{T}$ iff $\phi \in \sigma_i$.

Now, let me compare (IV.3.2) with the content of II.1. We see that the role played by v in the semantics of $\mathfrak{P}$ corresponds to that of σ in CPML. Therefore, the truth condition for $\Box\phi$ demands the quantification of σ in CPML if and only if it demands the quantification of v, which is the truth value assignment. This shows that CPML includes PLLM by (III.2.1). The reason is as follows. From (IV.1.2.4) and (IV.1.2.5), it is inferred that ϕ is L-true iff ϕ holds in every state description in CPML. Further, ϕ holds in every state description iff ϕ is true in every truth value assignment v. This shows exactly the same logical property as (II.3.5).

CQML also shows the same problem. A comparison of (iv) of (IV.2.2.3) and (II.3.11) reveals a very trivial difference between them. (II.3.11) demands the truth condition for $\Box\phi$ to include the universal quantification of both model $\mathfrak{A}$ and assignment s, while (IV.2.1.3) demands the universal quantification of model M only, where every quantified M should have the same universe as $|M|=|M'|$. This trivial difference is attributed to Carnap's definition of semantics using a set of state descriptions. Carnap's state description "contains, for every atomic sentence, either this sentence or its negation, but not both, and no other sentences" (Carnap 1947, p. 9). Therefore, every $|M_i|$, which is inferred from each state-description σ_i, is the same. Except this, there is no difference between (iv) of (IV.2.2.3) and (II.3.11).

V. Kanger's Quantified Modal Logic (L*) and PLLM

The next system considered is Kanger's (1957) quantified modal logic system. Kanger attempted to provide a language of quantified modal logic with model-theoretic semantics à la Tarski (1944) (Lindström 1998, p. 206). For this purpose, Kanger began from model-theoretic semantics of the first-order logic and extended it to L* that had modal operators. Here, Kanger's basic idea to construct L* is almost the same as that discussed in II. His system L* follows the Gentzen type to construct the first-order logic with which many current logicians may not be familiar. The main part of the system can be summed up as follows:

V.1. Kanger's Predicate Language System $\mathcal{L}$

(V.1.1) [Symbols, wffs][xiii] The set of primitive symbols and wffs are the same as that of $\mathcal{L}$.

(V.1.2) [Domain, frame][xiv] A nonempty set of objects $\mathcal{D}$ is the domain. And r such that $r \subseteq \mathcal{D}$ is the *frame*.

(V.1.3) [Primary valuation V] The primary valuation V is any binary operation V such that

(V.1.3.1) the domain of values of the first argument variable of V is the class of frames,

(V.1.3.2) the domain of values of the second argument variable of V is the class having terms, propositional constants, and n-ary predicates.

(V.1.3.3) $V(r,P) = \mathrm{T}$ or F. (Here, P is a propositional variable.)

(V.1.3.4) $V(r,F^n)$ is a subset of the Cartesian product of $r\,(r^n)$. (Here, F^n is an n-ary predicate.)

(V.1.3.5) $V(r,t)$ is an element of r. (Here, t is a term.)

When we say that a primary valuation V' is like V except for a (we write $V' =_a V$ in symbols), we mean that $V'(r,x) = V(r,x)$, except when $x = a$.

A primary valuation V is normal, if it holds for each r such that an ordered set $< v_1 \cdots v_n w >$ of elements of r is a member of $V(r,"\in")$ iff $< v_1 \cdots v_n >$ is a member of $w\,(n = 1,2,\cdots)$.

(V.1.4) [Secondary valuation T] A secondary valuation T is a certain trinary(3-nary) operation T. Given a frame and a primary valuation, T assigns **T** or **F** to each formula and sequent. The definition is as follows:

(V.1.4.1) The range of T is the class { T, F }.

(V.1.4.2) The domain of values of the first argument variable of T is the class of r.

(V.1.4.3) The domain of values of the second argument variable of T is the class of V.

(V.1.4.4) The domain of values of the third argument variable of T is the class of wffs.

(V.1.4.5) $T(r,V,P) = V(r,P)$.

(V.1.4.6) $T(r,V,F^n t_1 \cdots t_n) = \mathrm{T}$ iff $< V(r,t_1),\cdots,V(r,t_n) > \in V(r,F^n)$.

(V.1.4.7) $T(r,V,\neg\phi) = \mathrm{T}$ iff $T(r,V,\phi) = \mathrm{F}$.

(V.1.4.8) $T(r,V,(\phi \rightarrow \psi)) = \mathrm{F}$ iff $T(r,V,\phi) = \mathrm{T}$ and $T(r,V,\psi) = \mathrm{F}$.

(V.1.5) [System] A system $\mathfrak{s}$ refers to an ordered pair $< r, V >$.[xv] A statement ϕ is true in $\mathfrak{s} = < r, V >$ if $T(r,V,\phi) = \mathbf{T}$. Otherwise, ϕ is false.

(V.1.6) [Logical truth and validity] A sentence ϕ is *logically true in* r and *valid in* r if ϕ is true in $< r, V >$ for each V. If ϕ is true in every system, we say that ϕ is *logically true* and that ϕ is *valid* at the same time.

V.2. Kanger's Modal Logic System L∗

(V.2.1) [Syntax of L∗] Kanger extends L to L∗ by adding several clauses to the first-order language system L. First, he adds a set of one-place modal operators

$\Box_1, \Box_2, \cdots$[xvi]

to the set of symbols of L. He also adds some technical clauses inclusive of the one which is as follows: if ϕ is a wff of L^*, $\Box\phi$ is also a wff of L^*.

(V.2.2) [Valuation] Kanger defines the ordered quadruple "R_i ($i = 1, 2, \cdots$)" to be $< r', V', r, V >$. To state this intuitively, R_i is a relation of two systems $\mathfrak{s}$ and $\mathfrak{s}'$, for $< r, V >$ is a system $\mathfrak{s}$.

To obtain the valuation of L∗ Kanger extends the valuation for $\mathfrak{L}$ by adding the following clauses to the definition of T:

(V.2.2.1) $T(r,V,\Box_i\phi) = \mathbf{T}$ iff $T(r',V',\phi) = \mathbf{T}$ for each r' and V' such that $R_i(r',V',r,V)$ ($i = 1, 2, \cdots$).

(V.2.2.1) is the key point of the modal semantics of Kanger. This can be explained with the vocabulary of the Kripkean possible world semantics as follows: $\Box_i\phi$ is true in a system $< r, V >$ iff ϕ is true in any system $< r', V' >$ which is in the relation R_i with $< r, V >$. The interpretation of modal operator $\Box_i$ depends on the property of R_i. Hence, various properties of R_i, which are defined in the following clauses, classify the modalities of L∗.

(V.2.3) [Classification of the modalities][xvii]

(V.2.3.1) $\Box_i$ is an analytic necessity iff $\Box_i$ is regular and $R_i(r', V', r, V)$ always holds.

(V.2.3.2) $\Box_i$ is a logical necessity iff $R_i(r', V', r, V)$ always holds.

The definitions of analytic necessity and logical necessity in (V.2.3.1) and (V.2.3.2) can be summed up as follows:

(V.2.3.3) $T(r, V, \Box_N \phi) = \mathbf{T}$ iff $\forall r(T(r, V, \phi) = \mathbf{T})$

(V.2.3.4) $T(r, V, \Box_L \phi) = \mathbf{T}$ iff $\forall r \forall V(T(r, V, \phi) = \mathbf{T})$

V.3. PLLM in L∗ of Kanger

Let me repeat that Kanger's key idea of extending $\mathcal{L}$ and constructing L∗ is very similar to that of constructing $\mathcal{L}^{\#}$ in III. Even then, the occurrence of PLLM in L∗ and its cause are exactly the same as for $\mathcal{L}^{\#}$. I will examine how Lindström (1988, pp. 216–218) re-described Kanger's L∗.[xviii]

Lindström (1998) considers a first-order predicate language with an identity and a family of unary modal operators $\{\Box_i : i \in I\}$. Let me call this system KQML. Lindström transforms Kanger's system $< r, V >$ to the ordered pair $< D, v >$, where D is a domain and v is a valuation. Further, KQML has single $< D, v >$ exactly as $\mathcal{L}$ has single $\mathfrak{A}$. Let $|\mathfrak{A}| = D$ and $\mathfrak{A} = v$, then the semantics of $\mathcal{L}$ in II.2 can be defined by $< D, v >$. This is the common method for defining a first-order language.[xix] I do not believe that Lindström's understanding is wrong here. Then, L∗ simply has the model $< D, v >$ with a sentence ($\Box\phi$), whose truth condition cannot be provided by each $< D, v >$. That is, L∗ has the property of (III.2.1). As a matter of fact, it may not seem certain whether L or L∗ has a universe that does not have $\mathfrak{s} = < r, V >$ but a set of $\mathfrak{s}$, in some sense, according to Kanger's unique manner of defining the semantics. However, if you scrutinize (V.1.1)–(V.1.6), you will see that Kanger's L∗ only has a single system $\mathfrak{s} = < r, V >$.[xx]

Further, Lindström (1998) reforms the semantics of L∗ where the truth of wff ϕ is defined. According to his reformation, ϕ is defined to be "true in system $\mathfrak{s} = < D, v >$" in

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L*.

(V.3.1) $\mathfrak{s} \vDash P(t_1 \cdots t_n)$ iff $< v(D,t_1) \cdots v(D,t_n) > \in v(D,P)$.

(V.3.2) $\mathfrak{s} \vDash \neg\phi$ iff $\mathfrak{s} \nvDash \phi$.

(V.3.3) $\mathfrak{s} \vDash (\phi \rightarrow \psi)$ iff $\mathfrak{s} \nvDash \phi$ or $\mathfrak{s} \vDash \psi$.

(V.3.4) $< D,I,g > \vDash \forall x\phi$ iff $< D,I,g > \vDash \phi$, for each g', such that $g' =_x g$.

(V.3.5) For every operator □, $\mathfrak{s} \vDash \Box\phi$ iff $\forall \mathfrak{s}'$ and if $\mathfrak{s} R_{\Box} \mathfrak{s}'$, then $\mathfrak{s}' \vDash \phi$.

Kanger's L* contains various interesting ideas, but all these ideas are not relevant to my study of whether a formal semantics is constructed in the right manner. Hence, I end my discussion on Kanger at this point. Let me point out that the clause (V.3.5) shows the exact process of (II.3.11). This implies that L* has the property of (III.2.2). Hence, the truth condition of □ includes the quantification of system $\mathfrak{s}$ in (V.3.5), while the position of $\mathfrak{s}$ in L* is not different from that of $\mathfrak{A}$ in $\mathscr{L}^{\#}$. This is not because Lindström's (1998) reformation of Kanger is incorrect, but because Kanger's L* has PLLM.

If we understand the primary valuation V of (V.1.3), we can see that V is the same as the structure $\mathfrak{A}$, which assigns a reference to each syntactical symbol of $\mathscr{L}$, and r is the same as universe $|\mathfrak{A}|$ that generates the elements of ran($\mathfrak{A}$). This implies that the system $\mathfrak{s} = < r, V >$ is an interpretation that is the same as $< D, v >$ in $\mathscr{L}$. Further, if we understand the secondary valuation T of (V.1.4), we can see that T is a function that assigns **T** or **F** to wff, on the basis of $\mathfrak{s}$, which is composed of two terms (r and V).

Again, in summary, Kanger's L* has a symbol □ that represents the universal quantification of $\mathfrak{s}$, which corresponds to $\mathfrak{A}$ of $\mathscr{L}$, in a truth condition of some sentence. Moreover, it is clear that $\mathfrak{s}$ is a pure meta language that does not belong to ran(T) in L*. In particular, $r \in$ ran(V), but V ($\in \mathfrak{s}$) is a function whose domain includes the primitive symbols of syntax, i.e., interpretation. Hence, it is clear that Kanger's L* has the property of (III.2.1).

VI. Montague's MQML and PLLM

The approach of Montague (1960) to the quantified modal system is similar to that of

Kanger. Montague also has PLLM in his modal system as Kanger does. Like Kanger, Montague also begins from model-theoretic semantics of a non-modal first-order system, adds modal operators, and extends the system to a modal system MQML (Lindström 1998, p. 207). The detailed process is as follows:

VI.1. Montague's MQML System

(VI.1.1) [Model] The Montague model $\mathfrak{w}$ of language $\mathscr{L}$ is an ordered triple $<\mathscr{D}, \mathscr{i}, \mathscr{g}>$, where $\mathscr{D}$ is a nonempty set (the *domain*), $\mathscr{i}$ is a function that assigns appropriate denotations in $\mathscr{D}$ to the non-logical constants of $\mathscr{L}$, and $\mathscr{g}$ is a function that assigns values in $\mathscr{D}$ to the individual variables of $\mathscr{L}$, i.e., an assignment.

(VI.1.2) [Interpretation] An interpretation of language $\mathscr{L}$ is a definition of clause that "the Montague model $\mathfrak{w}$ satisfies formula ϕ." The definitions for a sentence or a formula that does not have the modal operator "□" follows the ways of the standard model-theoretic interpretation as suggested by Tarski (1944). An individual symbol is assigned the following value in a Montague model: for a term t, if t is an individual constant, $V_{\mathfrak{w}}(t) = \mathscr{i}(t)$, and if t is variable, $V_{\mathfrak{w}}(t) = \mathscr{g}(t)$.

(VI.2) On the basis of the previous definitions, Montague constructs recursive definitions for the formulas of MQML as follows:

(VI.2.1) If F^n is an n-any predicate and t_1, ⋯, t_n are terms, $\mathfrak{w} \vDash F^n t_1 \cdots t_n$ iff $<V_{\mathfrak{w}}(t_1) \cdots V_{\mathfrak{w}}(t_n)> \in \mathscr{i}$

(VI.2.2) $\mathfrak{w} \vDash \neg\phi$ iff $\mathfrak{w} \nvDash \phi$.

(VI.2.3) $\mathfrak{w} \vDash \phi \rightarrow \psi$ iff $\mathfrak{w} \nvDash \phi$ or $\mathfrak{w} \vDash \psi$.

(VI.2.4) $\mathfrak{w} \vDash \forall x\phi$ iff $\vDash \phi$ for every $\mathfrak{w}'$ such that $\mathfrak{w}Q\mathfrak{w}'$.

(VI.2.5) $\mathfrak{w} \vDash \Box_L \phi$ iff $\vDash \phi$ for every $\mathfrak{w}'$ such that $\mathfrak{w}L\mathfrak{w}'$.[xxi]

Now, taking into consideration (VI.2.4)-(VI.2.5), Q and L, which are relations between models, are defined as follows. Let $\mathfrak{w} = <\mathscr{D}, \mathscr{i}, \mathscr{g}>$ and $\mathfrak{w}' = <\mathscr{D}', \mathscr{i}', \mathscr{g}'>$. Then,

(VI.2.6) $\mathfrak{w}Q\mathfrak{w}'$ iff $\mathscr{D} = \mathscr{D}'$, $\mathscr{i} = \mathscr{i}'$, and $\mathscr{g}(\alpha) = \mathscr{g}'(\alpha)$ for every variable α different from "x."

(VI.2.7) $\mathfrak{w}L\mathfrak{w}'$ iff $\mathscr{D} = \mathscr{D}'$ and $\mathscr{g} = \mathscr{g}'$.

VI.2 PLLM in MQML

In Montague's definitions of truth conditions, (VI.2.4) and (VI.2.5) are consistently similar to each other. Hence, it might be confusing to believe that the definition of a truth condition for $\Box_L \phi$ has no problem as far as that of $\forall x\phi$ is correct because both are defined in a similar manner. Both the truth conditions define a certain relation of $\mathfrak{w}$ and $\mathfrak{w}'$ and describe the universal quantification of model $\mathfrak{w}$. However, if we scrutinize them, we can see that they are not the same in some key points. (VI.2.4) can be defined similarly without the quantification of interpretation, whereas (VI.2.5) cannot be.

Let me consider the truth condition for $\forall x\phi$ first. (VI.2.4) demands that ϕ be true in every $\mathfrak{w}'$; hence, it seems that the truth condition quantifies the models. However, by observing Q, which is defined as a relation of models in (VI.2.6), we can see that the abovementioned truth condition in fact quantifies only variable x. The difference between $\mathfrak{w}$ and $\mathfrak{w}'$ is only the value assignment for x, which is quantified by $\forall$. In contrast, from (VI.2.5) and (VI.2.7), we can infer that the truth condition for $\Box_L \phi$ quantifies $\mathcal{i}$ in $\mathfrak{w}=<\mathcal{D}, \mathcal{i}, \mathcal{g}>$. According to the definition of (VI.2.7), the difference between $\mathfrak{w}$ and $\mathfrak{w}'$ is $\mathcal{i}$. Here, $\mathcal{i}$ belongs to a pure meta language as it is a function that assigns the extension to each constant on the basis of $\mathcal{D}$. On the contrary, it can be said that the truth condition for $\forall x\phi$ quantifies a variable x, which belongs to an object language, while that for $\Box_L \phi$ quantifies $\mathcal{i}$, which belongs to a meta language. This shows exactly the same logical property as that of the PLLM system $\mathcal{CS}^{\#}$ discussed in (III.2.2).

We can easily observe that MQML has PLLM if we apply (III.2.1) to MQML. In the MQML model, $\mathfrak{w}$ is an ordered triple $<\mathcal{D}, \mathcal{i}, \mathcal{g}>$. This implies that an individual model $\mathfrak{w}$ cannot provide the truth condition for $\Box_L \phi$ by (VI.2.5) in MQML. Every $\mathfrak{w}'$ such that $\mathfrak{w}L\mathfrak{w}'$ should be considered in order to determine the truth value of $\Box_L \phi$.

VII. Conclusion

I have defined PLLM and discussed some important formal systems that have PLLM. The implication of this discussion is quite clear: As far as some formal systems include the PLLM that I have discussed in II, we cannot consider these systems to be formalized in the correct manner (at least for a part of the system).

Some people might suspect that my analysis is faulty because I have stated that the systems of Carnap, Kanger, and Montague, who are very influential in this field of study, contain PLLM. However, do consider the fact that these researchers identified necessity with "logical necessity" in their discussions. Carnap defines necessity as "L-truth," and explains that the concept of "L-truth" is explicatum of what Leibniz calls the necessary truth and Kant calls the analytic truth.[xxii] According to Lindström (1998), Kanger (1957) attempted a metalinguistic interpretation of modality, which is different from an object-level interpretation (Lindström 1998, pp. 213–216). Montague has treated the concerned subject in almost the same manner.

In summary, these philosophers attempted to construct formal semantics for modality by formalizing metalinguistic concepts (such as logical necessity). Such intentions considered, it would not be a surprising conclusion that they mixed a meta language with an object language in their systems, which led to the errors that I have suggested in II. Moreover, thus far, no one has discussed PLLM or a similar problem for formal systems.

With respect to the existing modal systems, the most influential modal semantics fortunately does not have PLLM. This semantics is the possible world semantics designed by S. Kripke. Historically, Kripke's possible world semantics were obtained with much difficulty, according to the studies of many scholars (Copeland 2002). In contrast, it seems that the modal systems that were analyzed as problematic in this paper could be obtained with relative ease by adding some simple clauses to the formal semantics for $\mathcal{L}$ suggested by Tarski (1944).[xxiii] I have reported the error that can creep in when such an easy process is used.

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i Therefore, $\mathcal{S}^*$ denotes $\mathcal{P}^*$ or $\mathcal{L}^*$ and $\mathcal{S}^{\#}$ denotes $\mathcal{P}^{\#}$ or $\mathcal{L}^{\#}$.

ii This section refers mainly to Enderton (1972).

iii I believe that this is the main reason that PLLM in the formal systems of some scholars could not be detected earlier.

iv On the left side, the entity next to $\models_{\mathfrak{A}}$ in the blank is the syntactical sentence.

v I believe that this ambiguity is completely irrelevant in this entire context.

vi This means that an element $\boldsymbol{m}$ of the universe $|\mathfrak{A}_n|$ is a model in the structure $\mathfrak{A}_n$ of $\mathcal{S}_1$. Therefore, structure $\mathfrak{A}_n$ of $\mathcal{S}_1$ is not different from another common structure $\mathfrak{A}$ of $\mathcal{L}$.

vii Enderton (1972) has an additional clause that an n−any operation is assigned to each n−place function symbol. However, I have omitted this clause in this paper.

viii It is very easy to see that the case of $f \in \mathrm{dom}(f)$ runs in the same manner.

ix This principle guarantees that the formal sentence and inference does not include an “implicit meaning.” This is the reason that we do not use common language but instead use a formal language.

x I am afraid that someone may criticize that getting rid of ϕs on both sides at once may lead to a confusion between the meta language and the object language. This may be correct, but the starting point of this problem is that the truth condition itself includes the mixture of a meta language and an object language.

xi As a matter of fact, CML includes an iota operator and a lambda operator as well. Hence, CML is larger than the sum of CPML and CQML.

xii The reason that $|M|=|M'|$ will be explained at the end of Ch. IV.

xiii Kanger defines the syntax in four steps: the symbols, formulas, statement, and sequents. More detailed parts of this definition seem irrelevant for my argument.

xiv The frame of Kanger is almost equivalent to the domain of the general model−theoretic semantics.

xv Kanger himself did not use the symbol “$\mathfrak{s}$” to denote a system. I introduced this symbol for the sake of an easy explanation.

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xvi Kanger's symbol for this is "M_i." I replaced this symbol with □ to aid in the readers' understanding.

xvii There are various modalities in L*. They are regular, ontological, purely ontological, absolute, realizable, realized, positively semi-iterative, negatively semi-iterative, iterative, analytically necessary, and logically necessary. It is not required to consider all these modalities; hence, only two of them, which are relevant to the topic discussed in this paper, are introduced.

xviii Lindström [1988], pp. 216-218.

xix Further, Lindström states that <D,v> can be understood as <D,I,g> if necessary, where D is the domain, I is the interpretation, and g is the assignment. v=I∪g. I have divided the assignment v into I and g in my consideration, because it is more convenient to compare the V (primary valuation) and the T (secondary valuation) of L* to the standard semantics of the first-order logic. That is, g is matched with V and I is matched with T.

xx Moreover, no one who has analyzed Kanger's L* has used a set of 𝔰 as the universe.

xxi Montague (1960) defines (VI.2.4) and (VI.2.5) as Q-satisfaction and L-satisfaction. Among his definitions, for example, L-satisfaction is

(ML') $\mathfrak{w} \models \Box_L \phi$ iff $\models \Box_L \phi$ for every $\mathfrak{w}'$, such that $\mathfrak{w} L \mathfrak{w}'$.

Further, Montague defines P-satisfaction, E-satisfaction, and so on. I have not discussed them in this paper.

xxii Carnap (1947), p. 8. interpreted necessity as validity that is a logical truth. Here, the validity is that which is in the meta language of the first-order language.

xxiii Compared with Kripke's system, the systems of Carnap, Kanger, and Montague are the results of an extremely trivial modification of the given semantics. However, they all produce the same effect and provide truth conditions for the modal sentences.

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